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LETTER TO THE EDITOR

# Short-range corrections to the order parameter of the Ising spin glass above the upper critical dimension

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**Abstract.** The expansion of the order parameter $q(x)$ of an Ising spin glass in the inverse number $1/z$ of interacting neighbours, corresponding to the standard field theoretic loop expansion, is considered above $d=6$ in the vicinity of the critical temperature. At first-loop order and in dimensions $d>8$, $q(x)$ is linear with an essentially temperature-independent slope and becomes a constant $q_1 \sim t$ ($t$ is the reduced temperature) beyond a breakpoint $x_1 \sim t$, just as in mean-field theory. For $t^{(d/2)-4}/z \ll 1$ the same holds true also in the range $6<d<8$, but in the fluctuation dominated regime $t^{(d/2)-4}/z \gg 1$ both the slope $c$ of $q(x)$ and the breakpoint acquire a non-trivial temperature dependence: $c \sim zt^{4-d/2}$ and $x_t \sim t^{d/2-3}/z$. Replica symmetry-breaking effects are found not only to be stable but even enhanced by fluctuations as $d$ decreases. The shifts in the exponents associated with $q(x)$ together with those in the exponents of the AT line, of the characteristic lengths, etc, observed previously by Green *et al* and by Fisher and Sompolinsky can all be incorporated into an effective mean-field theory valid for $6<d<8$. It is argued that higher-loop corrections cannot modify the exponents predicted by this effective mean-field theory.

Parisi's mean-field theory (MFT) [1] has now been widely accepted as the proper solution of the infinite-ranged (or infinite-dimensional) Sherrington–Kirkpatrick [2] spin-glass (SG) model. The problem of the finite-ranged spin glass in finite dimensions has remained controversial, however. On the one hand, the pheonomenological scaling approach, pioneered by McMillan [3] and extended by Fisher and Huse [4] and by Bray and Moore [5], has led to the conclusion that some of the most essential features of Parisi's MFT (like the existence of infinitely many, hierarchical organized phase space valleys, with the associated non-trivial order parameter $q(x)$, or persistence of the SG transition in an external magnetic field) will be absent in a finite-ranged model in any dimension $d<\infty$, or (according to Bray and Moore's less radical position) below $d=6$. On the other hand, recent evidence from a number of sources seems to point in the opposite direction: a $1/d$ expansion by Georges *et al* [6] shows that replica symmetry-breaking effects are actually enhanced with decreasing dimension. Monte Carlo work by Reger *et al* [7] finds a non-trivial distribution $P(q)$ of phase space distances in $d=4$, while Caracciolo *et al* [8] find strong indications for a non-trivial $q(x)$ in $d=3$ also in a field.

The powerful methods of field theory, namely the loop-expansion around MFT, offers a complementary alternative to the $1/d$ expansion. The building blocks of such an interacting SG field theory have been at our disposal for some time now: the complete

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set of free (Gaussian) propagators has been determined [9] and their physical meaning has been revealed [10] by expressing them in terms of overlaps of spin–spin correlation functions. It has been clear from the outset that the construction of the interacting field theory will not be an easy job: it was recognized, for example, that the anomalously strong infrared singularities in some of the free propagators lead to a paradox already in $d=6$ [10]. These high infrared powers were shown to be intimately linked to the small $x$ behaviour of $q(x)$ and, on the basis of a preliminary analysis of the one-loop equation of state, it was suggested that fluctuations may suppress $q(x)$ for small $x$ so allowing the theory to be continued to below $d=6$ [11].

Our purpose here is to examine the first-loop correction to $q(x)$ in detail. We work in the vicinity of the transition temperature throughout and in zero field in most of the paper. Also, we restrict ourselves to the dimensionality range $d>6$, i.e. above the upper critical dimension.

Our starting point is an Edwards–Anderson-like model [12] for a system of $N$ Ising spins on a $d$-dimensional lattice:

$$\mathcal{H}=-\sum_{(ij)} J_{ij}s_is_j \qquad s_i=\pm 1 \tag{1}$$

where the summation is over all pairs $(ij)$, but the interaction is finite ranged

$$J_{ij}=\frac{1}{\sqrt{z}}\tilde{J}_{ij}f\left(\frac{|\boldsymbol{r}_i-\boldsymbol{r}_j|}{\rho a}\right). \tag{2}$$

$\tilde{J}_{ij}$ are independent, Gaussian distributed random variables with zero mean and variance $\Delta$, $f$ is a smooth, positive cut-off function: $f(x)\sim 1$ for $x\leqslant 1$, and falls off sufficiently fast, say exponentially, for $x\gg 1$. $\rho$ is the range of the interaction in units of the lattice spacing $a$, while $z=\rho^d$ is the number of spins within the interaction radius. For $\rho\to\infty$ equation (1) goes over into the SK model, while, for $\rho$ large but finite, $1/z$ can be used as an expansion parameter to calculate corrections to the MF results.

The average over the random couplings (denoted by an overbar) is taken by the usual replica trick [12]. Through a number of formal steps [13] the $n$th power of the partition function can be expressed as a functional integral

$$\overline{Z^n}\sim\int[\mathrm{d}\phi]\,\mathrm{e}^{-\mathcal{L}}$$

over a set of fields $\phi^{\alpha\beta}$, where $\alpha$, $\beta$ are replica indices. Near the transition temperature it is sufficient to expand the Lagrangian $\mathcal{L}$ to quartic order and of the possible quartic couplings to keep only the one responsible for replica symmetry breaking:

$$\mathcal{L}=\tfrac{1}{2}\sum_{\alpha<\beta}\sum_{p}[(a\rho p)^2-2\tau]|\phi_p^{\alpha\beta}|^2-\frac{w}{\sqrt{N}}\sum_{\alpha<\beta<\gamma}\sum_{p_i}\delta^{Kr}_{p_1+p_2+p_3,0}\phi_{p_1}^{\alpha\beta}\phi_{p_2}^{\beta\gamma}\phi_{p_3}^{\gamma\alpha}$$
$$-\frac{u}{6N}\sum_{\alpha<\beta}\sum_{p_i}\delta^{Kr}_{p_1+p_2+p_3+p_4,0}\phi_{p_1}^{\alpha\beta}\phi_{p_2}^{\alpha\beta}\phi_{p_3}^{\alpha\beta}\phi_{p_4}^{\alpha\beta}. \tag{3}$$

Due to the finite range of the interaction the wavevector summations here are all restricted to $|\boldsymbol{p}|<1/\rho a$.

The coupling constants $w$, $u$, and the MF reduced temperature $\tau=(T_c^{MF}-T)/T_c^{MF}$ are related to the parameters of the original Hamiltonian (1) through the combination $K=(\Delta/k_BT)^2\sim 1$ as $w=K^3$, $u=K^4$ and $\tau=\tfrac{1}{2}(K^2-1)$, respectively, but with a slight generalization of the model we may regard $w$ and $u$ as essentially free (positive) parameters, while $\tau$ ($>0$) will be considered small.

Now the field is split into an equilibrium and a fluctuating part as

$$\phi_p^{\alpha\beta} = \sqrt{N}\, q_{\alpha\beta}\delta_{p,0}^{Kr} + \psi_p^{\alpha\beta}. \tag{4}$$

The equilibrium value $q_{\alpha\beta}$ is determined by the condition that the average of $\psi_p^{\alpha\beta}$ (taken with the weight $(\exp(-\mathscr{L}))$ vanish. To first order in $1/z$ this yields the following equation of state:

$$\begin{aligned} 2\tau q_{\alpha\beta} &+ w \sum_{\gamma\neq\alpha,\beta} q_{\alpha\beta}q_{\gamma\beta} + \tfrac{2}{3}uq_{\alpha\beta}^3 \\ &= -\frac{w}{z}\int_{|p|<1}\frac{\mathrm{d}^d p}{(2\pi)^d}\sum_{\gamma\neq\alpha,\beta} G_{\alpha\gamma,\gamma\beta}(p) - \frac{2u}{z}\int_{|p|<1}\frac{\mathrm{d}^d p}{(2\pi)^d}\, q_{\alpha\beta}G_{\alpha\beta,\alpha\beta}(p) \end{aligned} \tag{5}$$

where the propagators $G$ in the loop terms are given by

$$(G^{-1})_{\alpha\beta;\gamma\delta} = (p^2 - 2\tau - 2uq_{\alpha\beta}^2)\delta_{\alpha\beta,\gamma\delta}^{Kr} - w(\delta_{\alpha\gamma}^{Kr}q_{\beta\delta} + \delta_{\alpha\delta}^{Kr}q_{\beta\gamma} + \delta_{\beta\gamma}^{Kr}q_{\alpha\delta} + \delta_{\beta\delta}^{Kr}q_{\alpha\gamma}) \tag{6}$$

and a factor $a\rho$ has been absorbed into the momentum $p$.

Now let $\tau_c$ be that value of $\tau$ where the solution of (5) vanishes. This $\tau_c$ must be of order $1/z$. Since the loop terms are already $1/z$, $\tau$ can be replaced by the 'true' reduced temperature $t = \tau - \tau_c = (T_c - T)/T_c$ in the propagators on the RHS of (5). It is evident from (6) that in the limit $t$, $q_{\alpha\beta}\to 0$, $G_{\alpha\beta,\alpha\beta} = 1/p^2$ and $G_{\alpha\gamma,\gamma\beta} = wq_{\alpha\beta}/p^4$. Substituting these into (5) and dropping all terms higher than linear in $q_{\alpha\beta}$ we get the shift in the critical temperature:

$$\tau_c = \frac{T_c^{\mathrm{MF}} - T_c}{T_c^{\mathrm{MF}}} = \frac{1}{z}\int_{|p|<1}\frac{\mathrm{d}^d p}{(2\pi)^d}\left(\frac{w^2}{p^4} - \frac{u}{p^2}\right). \tag{7}$$

For reasonable values of the coupling constants (such as $w \sim u \sim 1$ corresponding to the original model) $T_c^{\mathrm{MF}} > T_c$, as expected.

Now we combine (5) and (7), and perform the replica summation by assuming that $q_{\alpha\beta}$ has a Parisi-like structure [1]:

$$2tq(x) - w\left(\int_0^x \mathrm{d}y\, q^2(y) + xq^2(x) + 2q(x)\int_x^1 \mathrm{d}y\, q(y)\right) + \frac{2}{3}uq^3(x) = L_w(x) + L_u(x) \tag{8}$$

where

$$L_w(x) = \frac{w}{z}\int_{|p|<1}\frac{\mathrm{d}^d p}{(2\pi)^d}\left[\int_0^x \mathrm{d}y\, G_{x1}^{yy}(p) + xG_{x1}^{xx}(p) + 2\int_x^1 \mathrm{d}y\, G_{x1}^{xy}(p) - \frac{2wq(x)}{p^4}\right] \tag{9}$$

$$L_u(x) = -\frac{2u}{z}\int_{|p|<1}\frac{\mathrm{d}^d p}{(2\pi)^d}\, q(x)\left[G_{11}^{xx}(p) - \frac{1}{p^2}\right]. \tag{10}$$

The parametrization for $G$ is as in [9].

In the MF limit $(z\to\infty)$ equation (8) is easily solved by repeated differentiation and leads to the well known result:

$$q^{\mathrm{MF}}(x) = \begin{cases} \dfrac{w}{2u}x & x < x_1^{\mathrm{MF}} = \dfrac{2ut}{w^2} + \mathrm{O}(t^2) \\[2ex] \dfrac{w}{2u}x_1^{\mathrm{MF}} = \dfrac{t}{w} + \mathrm{O}(t^2) & x > x_1^{\mathrm{MF}}. \end{cases} \tag{11}$$

The propagators in the loop terms $L_w$, $L_u$ can be obtained by inverting (6) which, under the Parisi parametrization, becomes a system of seven integral equations for $G$ as functional of $q(x)$. This system has been solved in the case $q(x)=q^{\mathrm{MF}}(x)$ in [9] and the RHS of (8), being of $O(1/z)$, this special solution is all we need in the following. (In fact, the solution [9] is given for $u=w=1$, the generalization is a matter of a trivial rescaling.)

The propagators obtained in [9] are the exact solutions of (6), so, as we have shown elsewhere [10], they contain all the information about the correlations of Gaussian fluctuations both inside a single phase space valley and between the different valleys. As such, they exhibit an extremely complicated dependence on the replica variables, but also on the momentum, having a markedly different behaviour in $p$, according to whether it is comparable to the 'large mass' $\sim t^{1/2}$ or to the 'small mass' $\sim t$ of the theory [14], or larger (smaller) than both. We have to realize, however, that most of this vast amount of information is not actually needed in the present context.

As long as $q(x)$ can be calculated perturbatively at all, it must stay close to its MF form (11), i.e. it must have a linear piece for small $x$, followed by a plateau beyond a breakpoint. In order to find such a solution, we need the loop terms in (8) to $O(x^3)$ only. Expanding $L_w$ and $L_u$ to that order we find that the constant and the $x^2$ terms vanish, as they should indeed, if a solution with the said properties is to exist. The coefficients of the $x$, $x^3$ terms are still far too complicated to be displayed here. As the next step towards rendering the job manageable, we note that working above the upper critical dimension, i.e. for $d>6$, one does not expect the far infrared region to give any significant contribution. Given the unconventional nature of the spin-glass propagators we felt nevertheless compelled to check this point and found that the contribution of the range where the momentum is comparable to or smaller than the small mass, $p^2 \leqslant t^2 \ll 1$, is indeed quite negligible. Therefore, we are certainly permitted to expand the $x$ and $x^3$ terms in the loops by assuming that $p$ is comparable to the large mass, $p\sim t^{1/2}$, or larger, and set an infrared cut-off, lying somewhere between the small and large mass scales, to the loop integrals in (9), (10). As a matter of fact, standard power counting tells us that above the upper critical dimension the region near the large mass should not normally matter either; all the relevant contributions should come from the vicinity of the upper cut-off. Working out the expansion we find that this is indeed so for all the terms except for a single one which scales as $p^{-8}$ and hence does pick up important contributions from the region near $p\sim t^{1/2}$ when we go below $d=8$. Apart from this particular term even the large mass can be neglected compared with $p^2$, which then leads us to the following simple forms:

$$L_w(x)=\frac{1}{z}\int_{|p|<1}\frac{\mathrm{d}^d p}{(2\pi)^d}\left\{2\frac{w^3}{u}\frac{xt}{p^6}+\frac{1}{3}\frac{w^5}{u^2}\frac{x^3}{p^6}-\frac{w^7}{u^3}\frac{x^3}{p^4(p^2+2t)^2}\right\} \tag{12}$$

$$L_u(x)=\frac{1}{z}\int_{|p|<1}\frac{\mathrm{d}^d p}{(2\pi)^d}\left\{-2w\frac{xt}{p^4}-\frac{1}{2}\frac{w^3}{u}\frac{x^3}{p^4}+\frac{1}{2}\frac{w^5}{u^2}\frac{x^3}{p^6}\right\}. \tag{13}$$

Terms such as $xt^2$ or $x^3t$, etc, which do not affect the leading temperature dependence of the solution, have all been dropped here, except in the 'dangerous term', the last in $L_w$, discussed above.

Let us now assume that $d$ is above, not only 6, but 8. Then the large mass can be discarded also in the dangerous term, and a solution of the form

$$q(x)=\begin{cases} cx & x<x_1 \\ cx_1 & x>x_1 \end{cases} \tag{14}$$

can be readily found by, e.g., simple substitution into (8). Collecting terms to $1/z$ the slope works out to be

$$c=\frac{w}{2u}\left[1+\frac{12K_d}{z}\left(-\frac{w^4}{u}\frac{1}{d-8}+\frac{5}{6}\frac{w^2}{d-6}-\frac{1}{2}\frac{u}{d-4}\right)\right] \qquad d>8 \tag{15}$$

where $(2\pi)^d K_d = S_d$ is the surface of the $d$-dimensional unit sphere, i.e. $2\pi^{d/2}/\Gamma(d/2)$. For the breakpoint we get

$$x_1=\frac{2ut}{w^2}\left[1+\frac{12K_d}{z}\left(\frac{w^4}{u}\frac{1}{d-8}-\frac{w^2}{d-6}+\frac{2}{3}\frac{u}{d-4}\right)\right] \qquad d>8. \tag{16}$$

Finally, the maximum of $q(x)$ can be obtained simply from $q_1 = cx_1$:

$$q_1=\frac{t}{w}\left[1-\frac{2K_d}{z}\left(\frac{w^2}{d-6}-\frac{u}{d-4}\right)\right] \tag{17}$$

valid for $d>6$.

For $d>8$ the results above are as expected: mean-field terms plus well-behaved corrections of $O(1/z)$. Moreover, the correction to the slope is negative, while that to the breakpoint is positive for any $w$, $u>0$. This means that replica symmetry breaking is not only stable against fluctuations, but is actually enhanced by them as $d$ decreases, in complete agreement with the conclusion from the $1/d$ expansion [6]. Note also that while the leading $1/d$ corrections to MFT come from small-scale fluctuations, the $1/z$ expansion picks up the effect of the very long wavelength ones, so the two approaches are in a sense complementary and their mutual consistency lends further support to the conclusion above.

As we approach $d=8$ from above, the corrections to $c$ and $x_1$ blow up. The breakdown of the $1/z$ expansion at $d=6$ would be perfectly normal: 6 is the upper critical dimension of the model. Why the expansion should break down at $d=8$ demands explanation, however. It is very instructive at this point to go back to the definition of the propagators, solve the Dyson equations (6) by direct iteration for $p^2 \gg t \gg q_{\alpha\beta}^2$ (remember the replacement $\tau\to t$) to order $p^{-8}$, and drop subleading terms as earlier. In so doing one finds:

$$L_w=\frac{1}{z}\int_{t^{1/2}<|p|<1}\frac{\mathrm{d}^dp}{(2\pi)^d}\left\{\frac{8w^2q_{\alpha\beta}t}{p^6}+\frac{4uw^2q_{\alpha\beta}^3+2w^3\sum_{\gamma\neq\alpha,\beta}q_{\alpha\gamma}q_{\gamma\beta}}{p^6}-\frac{8w^4q_{\alpha\beta}^3}{p^8}\right\} \tag{18}$$

$$L_u=\frac{1}{z}\int_{t^{1/2}<|p|<1}\frac{\mathrm{d}^dp}{(2\pi)^d}\left\{-\frac{4uq_{\alpha\beta}t}{p^4}-\frac{4u^2q_{\alpha\beta}^3}{p^4}+\frac{4uw^2q_{\alpha\beta}^3}{p^6}\right\}. \tag{19}$$

The meaning of the various terms in (12), (13) can now be understood by a term-by-term comparison with (18), (19): they are the $1/z$ corrections to (the derivatives with respect to $q_{\alpha\beta}$ of) the two-, three- and four-point functions. The dangerous term in particular, is related to the four-point function at zero external momenta. It is evident that this quantity (the 'box graph') should be singular at $d=8$.

It must be clear that the effect is not a peculiarity of spin glasses only. Many-point functions at exceptional momenta (where power-counting arguments break down) can blow up in high dimensions in any theory. For example, the sixth derivative of the Helmholtz potential with respect to the magnetization in an ordinary $\varphi^4$ theory will be singular in $d=6$, i.e. above the upper critical dimension of that model. This six-point function is related to a higher nonlinear susceptibility. It will also directly enter physical quantities such as the ordinary susceptibility as an insertion in some higher-loop corrections, but then finite momenta flow through it and the singularity will be suppressed by phase space factors, so that it will not cause the susceptibility to blow up.

Similarly, the singular contribution to the four-point function in our spin glass model will not lead to a proliferation of singularities in higher orders. Therefore, if we find a way to incorporate its effect into $q(x)$ the corrections to this new solution will be small.

Let us now go into the dimensionality range $6<d<8$. The bare $uq^3(x)$ term on the LHS of (8) is now competing with the dangerous term $\sim z^{-1}w^4q^3(x)t^{(d/2)-4}$ coming from $L_w$ on the right. As long as $z$ is very large and $t$ not too small, the bare term wins, and in this temperature region the ordinary $1/z$ expansion will still work, with $(1/z)t^{(d/2)-4} \ll 1$ providing a kind of Ginzburg criterion for its limit of validity. If we go closer to $T_c$, however, the bare term becomes negligible compared with what was supposed to be a small correction. It is clear that under these circumstances the dangerous contribution to the four-point function must be absorbed into the mean-field part on the left of (8) and only the rest of the loop terms can be regarded as a correction. With this we define an effective mean-field theory having the same structure as the original one but with the replacement of the bare four-point coupling $u$ by

$$\begin{aligned}\tilde{u} &= u + 12\frac{w^4}{z}\int\frac{\mathrm{d}^d p}{(2\pi)^d}\,\frac{1}{p^4(p^2+2t)^2} \\ &= u + 12K_d\frac{w^4}{z}(2t)^{(d/2)-4}\int_0^\infty \mathrm{d}x\,\frac{x^{d-5}}{(x^2+1)^2} \qquad 6<d<8.\end{aligned} \tag{20}$$

The solution for $q(x)$ in this effective MFT will be of the same form as (11) with $u$ replaced by $\tilde{u}$ everywhere. Hence in the fluctuation-dominated regime, where $t^{d/2-4}/z \gg 1$, we find that the slope $c=w/2\tilde{u}$ of $q(x)$ is $c\sim zt^{4-d/2}$, that is formally of $\mathrm{O}(z)$ yet very small, while the breakpoint is $x_1 \sim \tilde{u}t \sim t^{(d/2)-3}/z$. The maximum of $q(x)$ is independent of $u$, so it remains $q_1 \sim t$ as in ordinary MFT. A word of caution may be in order here: in the predictions of the effective MFT $d$ must not be taken to be exactly 6. At $d=6$ the other terms in $L_w$ and $L_u$ discarded in the effective MFT will also become important and produce the usual logarithms appearing at the upper critical dimension. We wish to discuss these and the region $d<6$ in a separate publication; for our present purposes $d$ must always be considered as maybe close to but definitely larger than 6.

Before deriving further conclusions from our effective MFT we have to clarify an all important point. We have seen how a dangerous correction to the four-point function enters the slope of $q(x)$ and modifies its temperature dependence for $6<d<8$. Compared with this, the other terms in (12), (13) represent small corrections that can be disregarded as far as the leading temperature dependence of $q(x)$ is concerned. It is obvious, however, that if we carry out the expansion of the loop integral to higher than cubic order in $q$ we shall generate corrections to the five-, six-, etc, point functions that blow up in higher and higher dimensions and, entering the higher Taylor coefficients of $q(x)$, threaten to destroy the linear solution described above. Now it is easy to see from (6) that the most dangerous type of terms that may appear on the right of the equation of state are of the form $q^k_{\alpha\beta}/p^{2k+2}$. It is quintessential that, as a detailed examination of the closed expressions of the propagators but also a direct expansion of the Dyson equation (6) demonstrates, the highest term of the above type that is actually allowed is the $q^3_{\alpha\beta}/p^8$ as in (18) that we have already included. Terms in $1/p^{10}$ do not contain $q^4_{\alpha\beta}$ but typically, e.g., $q^2_{\alpha\beta}\Sigma_\omega q_{\alpha\omega}q_{\omega\beta}$, which as an effect of ultrametric geometry is $\sim xq^4(x)$. More generally, for $k>3$, $q^k_{\alpha\beta}$ is substituted by combinations behaving as $x^l q^k(x)$, $l\geq 1$. Hence their effect is negligible.

The renormalization of the four-point function has far reaching consequences for the structure of the theory. With the replacement $u\to\tilde{u}$ our earlier results [14] concerning the fluctuation spectra can be readily extended to the present effective MFT: one can see at once that while the large masses remain of order $t^{1/2}$ as in ordinary MFT, the upper edge of the small mass band, $(2\tilde{u}q_1^2)^{1/2}$, becomes $\sim(w^2/z)^{1/2}t^{(d/4)-1}$. As $d$ approaches 6 this goes as $\sim t^{1/2}$, i.e. becomes of the same order in $t$ as the large mass! We are still left with infinitely many masses, of course, but instead of two bands scaling with different powers in temperature, both bands now scale with the same MF-like power. Similar effects are observed in other quantities as well. For example, the slope of the order parameter function becoming $c\sim t$ in the limit $d\to 6^+$ the whole function $q(x)$ becomes proportional to the reduced temperature which means that the various order-parameter combinations like $q_1$ or $\Delta=(q_1-\int_0^1 \mathrm{d}x\, q(x))/T$ which, above $d=8$, are $\sim t$ and $\sim t^2$, respectively, for $6<d<8$ behave as $q_1\sim t$ and $\Delta\sim t^{(d/2)-2}$ respectively, i.e. they scale with the same power when $d\to 6^+$. As a consequence of the shifts in various exponents, some scaling laws (e.g. $\beta\delta=1-\alpha/2+\gamma/2$), which are badly violated in the ordinary MFT of spin glasses, get gradually restored as $d$ approaches 6. The expert reader must have noticed that the scaling predictions of our effective MFT are all strictly parallel to those obtained by Fisher and Sompolinsky [15] on the basis of a simple two-parameter renormalization group argument. The use of a two-parameter RG above the upper critical dimension, i.e. in the context of a non-renormalizable theory with its infinitely many renormalization constants, is questionable, however, and this has cast a shade of doubt on their conclusions. We believe the justification for their procedure lies in the argument about the suppression of the singular higher-point vertices by powers of the replica variable $x\leq x_1\ll 1$, presented above.

To make further contact with earlier work we now consider the effect of a small external field $h$ on the order parameter $q(x)$. To this end we have to introduce a term $h^2$ on the LHS of (8), but, in principle, we should also modify the propagators in the loop terms. Now the dominant effect of the loop, the renormalization of the four-point coupling, can be shown to come from the term $\int_x^1 \mathrm{d}y\, G^{xy}_{x1}$ in $L_w$, and there from the interval $x_1<y<1$. From the formulae given in [9] one can easily see that the propagators are quite insensitive to a small field in that interval. Therefore the effective quartic coupling $\tilde{u}$ will be independent of $h$ and the well known results for the field dependence of $q(x)$ can be readily taken over from ordinary MFT. For small $x$ a lower plateau $q_0=(3h^2/4\tilde{u})^{1/3}$ will appear which, increasing with $h$, will reach $q_1=t/w$ on the Almeida–Thouless line [16] $h^2_{\mathrm{AT}}=4\tilde{u}t^3/3w^3$, i.e. $\sim t^3$ when $d>8$ and $\sim t^{(d/2)-1}$, when $6<d<8$ (in the fluctuation-dominated regime).

This coincides with the result found by Green *et al* [17] which they obtained from the zero of the replicon self-energy in a one-loop calculation in the disordered phase. The agreement between these two completely independent calculations is quite reassuring and demonstrates the consistency of our effective MFT. The shift of the AT exponent is another instance of the general trend of restoration of scaling on approaching $d=6$, as noted already by Fisher and Sompolinsky [15]. Green *et al* [17] suggested that the shifted AT exponent might be exact. This must indeed be true for all the exponents predicted by the effective MFT, because the singularity found in the four-point function does not proliferate, so the higher-loop corrections will not be more singular than the one-loop contribution worked out here.

We feel we now have a coherent picture of how the theory evolves with decreasing dimension towards a field theory which perserves the richness of Parisi's MFT [1] without its disturbing feature of having two length scales†. The restoration of scaling upon approaching the upper critical dimension from above raises the hope that renormalization group methods will enable one to penetrate the range $d < 6$. Work is in progress in this direction.

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† Note that although a single temperature scale is recovered, the (squared) masses still differ by a loop factor ($w^2/z$).